Research Statement

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My research lies in the fields of discrete differential geometry, which features the discretization of the theory of classical differential geometry. I am interested in the theoretical problems arising in this field, and its potential applications in computer science. I mainly worked on two projects during my graduate studies.

The first project investigated the convergence of the discrete unformization factors to the classical uniformization functions on closed surfaces. In [15], we proved that for a reasonable geodesic triangulation T on a Riemannian surface (M, g), the discrete uniformization mapping of its induced polyhedral metric approximates the uniformization mapping for (M, g) by an error in the order of the maximal edge length of edges in T. In [9], we proved a similar result for surfaces of genus zero.

In the second project, we studied the space of geodesic triangulations of closed surfaces. In [10], we proved the contractibility of the space of geodesic triangulations on a closed Riemannian surface of negative curvature by generalizing Tutte's embedding theorem. This solves a problem proposed by Connelly et al. [1] in 1983, in the case of hyperbolic surfaces. In [11], we also proved that the space of geodesic triangulations on flat tori is homotopically equivalent to the torus.

Now, I shall describe below the results in details and some problems I will work on in the near future.

1 Convergence of discrete unformization factors on closed surfaces

The Poincaré-Koebe uniformization theorem states that any simply connected Riemannian surface (M, g) is conformally equivalent to the unit sphere $\mathbb{S}^2$, the complex plane $\mathbb{C}$, or the open unit disk $\mathbb{D}$. As a consequence, any smooth Riemannian metric g on a connected surface M is conformally equivalent to a Riemannian metric $\tilde{g}$ of constant curvature 0 or ± 1.

There are analogus uniformization results in the discrete setting. Given a closed surface S and a finite non-empty set $V \subset S$, we call (S, V) a marked surface. A *triangulation* of a marked surface (S, V) is a triangulation of S so that its vertex set is V. A piecewise flat (hyperbolic) polyhedral metric on (S, V) is flat (hyperbolic) cone metric on S whose cone points are in V. The *discrete curvature* of a PL metric on (S, V) is the function on V sending a vertex $v \in V$ to 2π less the cone angle at V. Every piecewise flat (hyperbolic) metric has an associated *Delaunay triangulation* which has the property that the interior of the circumcircle of each triangle contains no other vertices in the universal cover of S. For a geodesic triangulation T on a Riemannian surface (M, g) with vertex set $V(T)$, it induces a piecewise polyhedral metric (T, l) on the marked surface $(M, V(T))$, where $l(e)$ is the geodesic length of an edge e measured in g. The discrete conformal class for piecewise polyhedral metric, first defined by Gu et al. [8],[7], is as in the following.

Definition 1.1. Two two piecewise flat (or hyperbolic) polyhedral metrics d and d' on a marked surface (S, V) are discrete conformal if there exists a sequence of sequence of flat (or hyperbolic) polyhedral metrics $d = d_1, d_2, \cdots, d_m = d'$ and triangulations $\mathcal{T}_1, \cdots, \mathcal{T}_m$ on (S, V) such that

(a) Each $\mathcal{T}_i$ is Delaunay in d_i.

(b) • If $\mathcal{T}_i = \mathcal{T}_{i+1}$, then for any edge $jk \in E(\mathcal{T}_i)$

$$l_{i+1}(jk) = e^{\frac{u_j+u_k}{2}} l_i(jk) \qquad \text{for piecewise flat metric}$$
$$\sinh(l_{i+1}(jk)/2) = e^{\frac{u_j+u_k}{2}} \sinh(l_i(jk)/2) \qquad \text{for piecewise hyperbolic metric}$$

for some vertex function $u : V \to \mathbb{R}$. Here we use u_i to express the value of u at vertex i. We denote above relations by $u * l$ and $u *_h l$ respectively.

• If $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then $(\mathcal{T}_i, l_i)$ is isometric to $(\mathcal{T}_{i+1}, l_{i+1})$ relative to (S, V).

In [8], Gu-Luo-Sun-Wu proved that every piecewise flat polyhedral metric is discrete conformal to a piecewise flat polyhedral metric with a given discrete curvature satisfying discrete Gauss-Bonnet formula. For the special case of precribled curvature $K^* = 2\pi\chi(S)/|V|$, it will give us a constant curvature PL metric, unique up to scaling, discrete to the given piecewise flat polyhedral metric. In [7], Gu-Guo-Luo-Sun-Wu proved the similar discrete conformality result for the piecewise hyperbolic polyhedral metric. Rivin's realization theorem for ideal hyperbolic polyhedra with prescribed intrinsic metric [12] implies that for any given piecewise flat metric on the sphere, there exists a polyhedron inscribed in the sphere inducing a piecewise flat metric discrete conformal to the prescribed piecewise flat metric.

Motivated by the Delaunay condition of a piecewise polyhedral metric, we define the $\epsilon-$regularity for a piecewise polyhedral metric denoted by (T, l).

Definition 1.2. A piecewise polyhedral metric (T, l) is called *ϵ-regular* if

(a) any inner angle $\theta^i_{ijk} \geq \epsilon$, and

(b) for any adjacent triangles $\triangle ijk$ and $\triangle ijl$, $\theta^k_{ijk} + \theta^l_{ijk} \leq \pi - \epsilon$,

where θ^i_{ijk} denotes the angle of triange $\triangle ijk$ at vertex i.

In the work by Colin de Verdiére [2], a family of strictly acute triangulations on any Riemannain surface with explicit bounds on angles are constructed, and maximal edge lengths of these acute triangulations approach zero. This implies the existence of the ϵ-regular geodesic triangulations on any Riemannian surface of arbitrary upper bound of edge length.

For simplicity, we will use $(T, l)_E$ and $(T, l)_H$ to denote the piecewise flat metric and hyperbolic metric respectively. Our result for the piecewise hyperbolic metric on closed surfaces of genus $g > 1$ is stated as follows:

Theorem 1.3. *[15](with Tianqi Wu) Suppose (M, g) is a closed orientable smooth Riemannian surface with genus > 1 with the unique uniformization factor $\bar{u} = \bar{u}_{M,g} \in C^\infty(M)$ such that $e^{2\bar{u}}g$ is hyperbolic. Then for any $\epsilon > 0$, there exist $\delta = \delta(M, g, \epsilon) > 0$ and $C = C(M, g, \epsilon)$ such that for any $\epsilon-$regular geodesic triangulation T of (M, g) with associated edge length $|l| < \delta$, then*

1. *there exists a unique discrete conformal factor $u \in \mathbb{R}^{V(T)}$, such that $(T, u *_h l)_H$ is globally hyperbolic, and*
2. *$\left|u - \bar{u}|_{V(T)}\right| \leq C|l|_\infty$.*

The theorems for the piecewise flat metric on torus and the piecewise spherical metric on sphere are smilar. For torus, we require the normalized area condition, i.e

Theorem 1.4. *[15](with Tianqi Wu) Suppose (M, g) is a closed orientable smooth Riemannian surface of genus 1 with the uniformization function $\bar{u} = \bar{u}_{M,g} \in C^\infty(M)$ is the unique uniformization conformal factor such that $e^{2\bar{u}}g$ is flat and $Area(M, e^{2\bar{u}}g) = 1$. Then for any $\epsilon > 0$, there exists $\delta = \delta(M, g, \epsilon) > 0$ and $C = C(M, g, \epsilon)$ such that for any ϵ-regular geodesic triangulation T of (M, g) with associated edge length $|l| < \delta$, then*

1. *there exists a unique discrete conformal factor $u \in \mathbb{R}^{V(T)}$, such that $(T, u * l)_E$ is globally flat and $Area((T, u * l)_E) = 1$, and*

2. $\left|u - \bar{u}|_{V(T)}\right| \leq C|l|_\infty$.

For the sphere case [9], we require that the uniformization mapping fixes three marked points.

Theorem 1.5. *(with Yanwen Luo and Tianqi Wu) Suppose (M, g) is a closed smooth Riemannian surface of genus zero with three marked points X, Y, Z, and $\bar{u} \in C^\infty(M)$ is the uniformization factor such that $(M, e^{2\bar{u}}g)$ is isometric to the unit sphere $\mathbb{S}^2 \cong \hat{\mathbb{C}}$ through a conformal map ϕ, and $\phi(Z) = 0$, $\phi(Y) = 1$, $\phi(X) = \infty$. Then for any $\epsilon > 0$, there exist $\delta = \delta(M, g, \epsilon) > 0$ and $C = C(M, g, \epsilon) > 0$ such that for any ϵ-regular triangulation T of (M, g) with the associated edge length $|l| \leq \delta$,*

1. *there exists a unique discrete conformal factor $u \in \mathbb{R}^{V(T)}$, such that $(T, u * \sin \frac{l}{2})_E$ is isometric to an Euclidean polyhedral surface inscribed in the unit sphere through a map ψ such that $\psi(Z) = 0$, $\psi(Y) = 1$, and $\psi(X) = \infty$;*

2. $|u - \bar{u}|_{V(T)}| \leq C|l|_\infty$.

2 The space of geodesic triangulations on surfaces

In [11] and [10], we study the space of geodesic triangulations of a surface within a fixed homotopy type. Such space can be viewed as a discrete analogue of the space of surface diffeomorphisms homotopic to the identity. Smale [13] proved that the group of diffeomorphisms of a closed 2-disk fixing boundary pointwisely is a contractible space. Earle and Eells [3] identified the homotopy types of the topological groups of all orientation-preserving diffeomorphisms homotopic to the identity for any closed surfaces. Our theorem 2.1 and 2.2 can be viewed as the discrete analogue for their results.

I will state the results for space of geodesic triangulations on Riemannian surface of non-positive Gaussian curvature first. Assume (M, g) is a closed connected orientable smooth Riemannian surface of non-positive Gaussian curvature. A topological triangulation $T = (V, E, F)$ of M is a marking homeomorphism ψ from the polyhedron $|T|$ of the simplicial complex T with sets of vertices V, edges E, and faces F to M. For convenience, we label the vertices as $1, 2, ..., n$ where $n = |V|$. The edge in E determined by vertices i and j is denoted as ij.

Let T^1 be the 1-skeleton of T, and denote $X = X(M, T, \psi)$ as the set of all geodesic triangulations homotopic to $\psi|_{T^1}$. More specifically, X contains all the embeddings $\varphi : T^1 \to M$ satisfying that

(a) the restriction φ_{ij} of φ on each edge ij is a geodesic parametrized by constant speed, and

(b) φ is homotopic to the restriction of ψ on $T^{(1)}$.

Then our result can be stated as follows.

Theorem 2.1. *[10](with Yanwen Luo and Tianqi Wu) For a closed orientable Riemannian surface (M, g) of negative curvature, $X(M, T, \psi)$ is contractible. In particular, it is connected.*

Now we move on to the case of flat tori. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = [0, 1]^2/\sim$ be the flat torus constructed by gluing the opposite sides of the unit square in $\mathbb{R}^2$.

Here we use the similar set up as in the case of Riemannian surface of non-positive curvature. Let $X = X(\mathbb{T}^2, T, \psi)$ replace $X(M, T, \psi)$. The result for the case of flat tori is stated as follows.

Theorem 2.2 ([11]). *(with Yanwen Luo and Tianqi Wu) Given a topological triangulation (T, ψ) of flat tori $\mathbb{T}^2$, $X(M, T, \psi)$ is homotopically equivalent to a torus.*

For any geodesic triangulation $\varphi \in X$ we can always translate φ on $\mathbb{T}^2$ to make the image $\varphi(v_1)$ of the first vertex v_1 be at the (quotient of the) origin $(0,0)$. By this normalization, we can decompose X as $X(\mathbb{T}^2, T, \psi) = X_0 \times \mathbb{T}^2$, where

$$X_0 = X_0(\mathbb{T}^2, \mathcal{T}, \psi) = \{\varphi \in X : \varphi(v_1) = (0,0)\}.$$

Since there are affine transformations between any two flat tori, and an affine transformation always preserves the geodesic triangulations, we can reduce theorem 2.2 to the following.

Theorem 2.3. *Given a topological triangulation $(\mathcal{T}, \psi)$ of $\mathbb{T}^2$, the space $X_0 = X_0(\mathbb{T}^2, \mathcal{T}, \psi)$ is contractible.*

The idea for the proofs of theorem 2.1 and 2.3 are similar. I will sketch the proof of theorem 2.1 in the rest of this section.

Let $\tilde{X} = \tilde{X}(M, T, \psi)$ be the super space of $X = X(M, T, \psi)$, containing all the continuous maps $\varphi : T^1 \to M$ satisfying that

(1) The restriction φ_{ij} of φ on the edge ij is a geodesic, and

(2) φ is homotopic to $\psi|_{T^1}$.

An element in $\tilde{X}$ is called as a geodesic mapping.

Denote $\vec{E}$ as the set of directed edges of triangulation T, and a directed edge starting from the vertex i ending at the vertex j is denoted as (i,j). A vector $w \in \mathbb{R}^{\vec{E}}_{>0}$ is called a *weight* of T. For any weight w and a geodesic mapping $\varphi \in \tilde{X}$, we say φ is w-balanced if for any $i \in V$,

$$\sum_{j:ij \in E} w_{ij} \vec{v}_{ij} = 0,$$

where $\vec{v}_{ij} \in T_{q_i}M$ is defined with the exponential map $\exp : TM \to M$ such that $\exp_{q_i}(t\vec{v}_{ij}) = \varphi \circ e_{ij}(t)$ for $t \in [0,1]$.

The key ingredient for the proof of theorem 2.1 is to generalize Tutte's embedding theorem [14] to closed surfaces with negative curvature. Specifically, we prove the following two theorems.

Theorem 2.4. *Assume the Riemannian metric g on M has strictly negative curvature. For any weight w, there exists a unique $\varphi \in \tilde{X}(M, T, \psi)$ that is w-balanced. Denote such φ as $\Phi(w)$, and then Φ is a continuous map from $\mathbb{R}^{\vec{E}}_{>0}$ to $\tilde{X}$.*

Theorem 2.5. *If $\varphi \in \tilde{X}$ is w-balanced for some weight w, then $\varphi \in X$.*

Theorem 2.4 consists of three parts: the existence of w-balanced geodesic mapping for all $w \in \mathbb{R}^{\vec{E}}_{>0}$, the uniqueness of w-balanced geodesic mapping, and the continuity of Φ. Theorem 2.5 implies that a w-balanced geodesic mapping is a geodesic triangulation.

In the oppositie direction, we can construct a weight w for a geodesic embedding $\varphi \in X$, using *mean value coordinates* by Floater [5]. Given $\varphi \in X$, mean value coordinates are defined to be

$$w_{ij} = \frac{\tan(\alpha_{ij}/2) + \tan(\beta_{ij}/2)}{|\vec{v}_{ij}|},$$

where $|\vec{v}_{ij}|$ equals to the geodesic length of $\varphi \circ e_{ij}([0,1])$, and α_{ij} and β_{ij} are the two inner angles in $\varphi(T^{(1)})$ at the vertex $\varphi(i)$ sharing the edge $\varphi \circ e_{ij}([0,1])$. See Figure 1. The construction of mean value coordinates gives a continuous map $\Psi : X \to \mathbb{R}^{\vec{E}}_{>0}$. Furthermore, a geodesic embedding $\varphi \in X$ is $\Psi(\varphi)$-balanced by Floater's mean value theorem [5]. Namely, we have a section $\Phi \circ \Psi = id_X$. Then theorem 2.1 is a direct consequence of theorems 2.4 and 2.5.

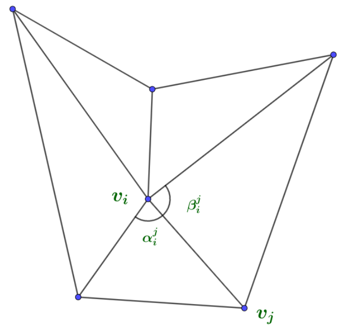

Figure 1: Mean value cooridinate

3 Future work

3.1 Graph Morphing Problem

There is a natural relation between the connectivity of spaces of geodesic triangulations and the graph morphing problem on surfaces. The connectivity implies that it is possible to construct a path connecting any two geodesic triangulation. In other words, there exists a morphing between two different geodesic embeddings of the 1-skeleton of the triangulation. A practical algorithm to compute this morphing has potential applications in the field of computational geometry.

Floater and Gotsman [6] proposed a framework to compute a morphing between two given geodesic triangulations of a convex polygon. They constructed a section $\sigma : X(\mathcal{P}, \mathcal{T}) \to W(\mathcal{P}, \mathcal{T})$. Then two geodesic triangulations are lifted to two weights in the space of weights, which is a convex subset of the Euclidean space. Hence, the linear interpolation between the two weights projects to a morphing between the two geodesic triangulations.

This framework has been applied to construct morphings on flat tori. As mentioned above, the Tutte map can not be defined for all positive weights. Erickson and Lin [4] defined the space of *morphable* weight in place of $W(\mathcal{P}, \mathcal{T})$ so as to solve the graph morphing problem.

Since we showed that the Tutte map is well-defined for all the weights if the surface has negative curvature. So we are interesting in the following question.

Question 1. *Can we design an algorithm to construct morphings on a surface of negative curvature ?*

Also, the proof of the existence and uniqueness theorem of a geodesic triangulation of a given weight is not constructive. Hence, it is interesting to explore to following question.

Question 2. *Can we design a variational method by minimizing the residue vectors as a discrete version of the Dirichlet energy to compute the embedding corresponding to any weight ?*

3.2 Homotopy type of space of geodesic triangulations of unit sphere

In [10] and [11], we have shown that for the flat tori and hyperbolic surfaces, the space $X(M, T, \psi)$ is homotopic equivalent to the group of its orientation-preserving isometries isotopic to the identity. Specifically,

- $X(\mathbb{T}^2, T, \psi)$ is homotopic equivalent to $\mathbb{T}^2$.
- $X(S_g, T, \psi)$ is contractible.

where S_g is a closed surface of genus $g > 1$. The next question we want to work on is:

Question 3. *Is $X(\mathbb{S}^2, T, \psi)$ homotopic equivalent to $SO(3)$?*

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